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BDDC for Mixed-Hybrid Formulation of Flow in Porous Media with Combined Mesh Dimensions

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Abstract: We extend the Balancing Domain Decomposition by Constraints (BDDC) method to mixed-hybrid finite element discretization of flow in porous media with combined mesh dimensions. It is assumed that the modeler desires to incorporate into the finite element model a detailed description of the reservoir geometry, including a possibly complex structure of vugs, cavities and fractures with various topology and sizes. Use of the mixed-hybrid formulation is motivated by an effort to modify the saddle-point problem into the one which is symmetric, positive definite, so that conjugate gradients with the BDDC preconditioner can be used. The preconditioner then takes an advantage of the special structure of the global finite element matrix. We apply our solver to several academic and real world problems, and the numerical experiments indicate that our approach allows for an efficient and scalable iterative solution of the resulting large systems of equations.

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1. Introduction

A mathematical description of flow in porous media is essential for building models with applications in, e.g., water management, oil and gas recovery, carbon dioxide (CO2) sequestration or nuclear waste disposal. In order

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to set up a reliable geomodel, one needs to have a good knowledge of the reservoir geometry and input parameters. The parameters include mainly an identification of the effective permeability tensor, and locations and intensities of sources and sinks that drive the right-hand side. However, our main focus here will be on a detailed description of the computational domain and its geometry into the model setup. For example, the flow of water in granite rock, which represents one of the suitable sites for nuclear waste deposit, is conducted by the complex system of vugs, cavities and fractures with various topology and sizes. These alter the effective permeability and therefore should be accurately accounted for in the geomodel. A unified approach to modeling of free-flow and porous media region can be provided by the so called Stokes-Brinkman equation, which reduces to either Stokes or Darcy model in certain parameter limits within the Multiscale mixed finite-element (MsMFE) framework [15]. Alternatively, the small free-flow regions can be incorporated into Darcy’s model using homogenization or upscaling techniques see, e.g., an overview [8]. Nevertheless, the preferential flow in large geological dislocations and their intersections should be considered as two- and one-dimensional flows, respectively. Due to the complex structure of the domains, it might be advantageous to use finite element methods (FEM). The finite element meshes are unstructured, and they also combine different spatial dimensions (linear in 1D, triangular in 2D, and tetrahedral in 3D). We assume that the fracture are filled with debris and therefore the Darcy’s law can be applied as well, see [21] for a related approach. The systems of linear equations obtained by the FEM discretization are typically very large and use of direct solvers might be prohibitive. The systems are also typically ill-conditioned, due to mixing of spatial dimensions and permeability coefficients, and therefore they might be challenging for iterative solvers as well. The discretization leads to matrices with a well known saddle-point structure

$$\begin{bmatrix}
A & B^T \\
B & -tC
\end{bmatrix}, \tag{1}
$$

where $$A$$ is symmetric and positive definite on the kernel of $$B$$, $$C$$ is symmetric and positive semi-definite, and $$t \geq 0$$ is a parameter in our case equal either to 0 or 1. The former presents the standard saddle-point case, while the latter arises when fractures are modeled. The iterative solution of systems with this structure is a frequently studied topic, see, e.g., [9, 11, 32, 33] or the monographs [31, 38, Chapters 9] and the reference therein.

Here we use an approach based on reduction. We eliminate the block $$A$$ (and a “bit” more) and introduce a symmetric and positive definite Schur complement. Our main interest is the iterative solution of the reduced system, and in particular, using preconditioned method of conjugate gradients (PCG). Our preconditioner is a variant of the Balancing Domain Decomposition by Constraints (BDDC), which is currently one of the most popular methods of iterative substructuring. It has been proposed independently by Cros [7], Dohrmann [10], and Fragakis and Papadrakakis [13], see [19, 29] for a proof of equivalence. Even though the BDDC has been aimed originally for elliptic problems, it has been successfully extended beyond elliptic problems, e.g., in [17, 37] and to multiple levels [20, 34, 35]. An optimal setup of the BDDC has been studied, e.g., in [16, 18, 26, 28] and the most recently in [30]. Here we are interested in an application of the BDDC method to the flow in fractured
porous media, which is a class of saddle-point problems (1). When \( t = 0 \) in (1), one approach is to retain the original \emph{primal} variables and use an algebraic trick of Ewing and Wang [12], cf. also Mathew [25]. Such approach to the BDDC has been presented by Tu [32], and it has been recently also extended into multiple levels [27, 36]. However we will favor an alternative, \emph{dual} or respective \emph{hybrid}, approach.

In this paper, we extend the BDDC to mixed-hybrid formulation of subsurface flow problems with combined mesh dimensions. The mixed-hybrid formulation is motivated by an effort to modify the saddle-point problem, obtained by the mixed finite element formulation, into the one which is symmetric, positive definite, so that conjugate gradients with the BDDC preconditioner can be used. Our approach is based on Method 2 from [14] described therein for the case of two substructures. That method was extended for the case of many substructures in [6], preconditioned by BDD in [5], and by the BDDC in [33]. Our main focus is on accommodating the BDDC solver to flows in porous media with combined mesh dimensions. We take advantage of the special structure of the blocks in the matrix (1) studied in detail in [22–24], and in particular by Březina and Hokr [3] who considered a nonzero structure of the block \( C \) resulting from a combination of meshes with different spatial dimensions.

The paper is organized as follows. In Section 2 we introduce the model problem and its mixed finite element formulation. In Section 3 we introduce the substructuring components and derive the mixed-hybrid formulation of the problem. In Section 4 we present the BDDC preconditioner. In Section 5 we discuss the details of modelling fractured porous media and combining meshes of different dimensions. In Section 6 we discuss details of our parallel implementation, and finally in Section 7 we report on numerical experiments.

For simplicity we do not distinguish notation of finite element functions and corresponding degrees of freedom, neither of linear operators and matrices within a specific basis – the meaning is clear from the context. The transpose is denoted by \( A^T \) and the energy norm of a symmetric positive definite matrix \( A \) by \( \| x \|_A = \sqrt{x^T A x} \).

## 2. Model problem

Let \( \Omega \) be an open bounded polygonal domain in \( \mathbb{R}^3 \). We are interested in a solution of the following problem combining the Darcy’s law and the equation of continuity in the mixed form as

\[
\begin{align*}
\mathbf{k}^{-1} \mathbf{u} + \nabla p &= 0 \quad \text{in } \Omega, \\
\nabla \cdot \mathbf{u} &= f \quad \text{in } \Omega, \\
p &= g_N \quad \text{on } \Gamma_N, \\
\mathbf{u} \cdot \mathbf{n} &= g_E \quad \text{on } \Gamma_E,
\end{align*}
\]

subjected to boundary conditions on \( \partial \Omega = \Gamma_N \cup \Gamma_E \), with \( \mathbf{n} \) as the unit normal defined almost everywhere on \( \partial \Omega \), such that \( g_N \in H^{1/2}(\Gamma_N), \ g_E \in H^{1/2}_{00}(\Gamma_E) \), but without loss of generality we will consider \( g_E = 0 \). We assume that the tensor of hydraulic conductivity \( \mathbf{k} \) is symmetric, positive definite, and \( f \in L^2(\Omega) \). The variable \( \mathbf{u} \) describes the flux and \( p \) the pressure in an aquifer \( \Omega \), cf., e.g., [1, 4] for discussion of application background.
Let us define a space
\[ H(\Omega; \text{div}) = \{ v : v \in L^2(\Omega); \nabla \cdot v \in L^2(\Omega); \text{ and } v \cdot n = 0 \text{ on } \partial \Omega \cap \Gamma_E \}, \]
and denoting by \( H_\Omega \) the diameter of the domain \( \Omega \), we will equip \( H(\Omega; \text{div}) \) with the norm
\[ \| v \|_{H(\Omega; \text{div})}^2 = \| v \|_{L^2(\Omega)}^2 + H_\Omega^2 \| \nabla \cdot v \|_{L^2(\Omega)}^2. \]

Let \( V \) and \( Q \) be the lowest order Raviart-Thomas (RT0) mixed finite element spaces defined by the triangulation \( \mathcal{T}_h \) of \( \Omega \), where \( h \) denotes the mesh size. We refer, e.g., to the monograph [2] for detailed description of mixed finite elements, and only note that \( V \) and \( Q \) are finite dimensional subspaces of \( H(\Omega; \text{div}) \) and \( L^2(\Omega) \), respectively.

In the mixed finite element approximation of problem (2)-(5) we look for a pair \( \{ u, p \} \in V \times Q \) that satisfies
\begin{align*}
\int_\Omega k^{-1} u \cdot v \, dx - \int_\Omega p \nabla \cdot v \, dx &= - \int_{\Gamma_D} g_N \cdot v \, n \, ds \quad \forall v \in V, \tag{6} \\
- \int_\Omega \nabla \cdot u \, q \, dx &= - \int_\Omega f q \, dx \quad \forall q \in Q. \tag{7}
\end{align*}

The mixed-hybrid formulation, introduced in the next section, is motivated by an effort to modify the saddle-point problem (6)-(7) into the one which leads to symmetric, positive definite matrices.

### 3. Substructuring and the mixed-hybrid formulation

Let \( \Omega \) be decomposed into nonoverlapping subdomains \( \Omega^i, i = 1, \ldots, N \), also called substructures, forming a quasi-uniform triangulation of \( \Omega \) with characteristic subdomain size \( H \). Let \( \Gamma^i \) be the set of boundary degrees of freedom on \( \partial \Omega^i \setminus \partial \Omega \) and let \( \Gamma = \bigcup_{i=1}^N \Gamma^i \). Let us denote by \( \mathcal{F} \) the set of all faces between substructures, i.e., in the present context the set of all intersections \( \Gamma^{ij} = \Gamma^i \cap \Gamma^j \), such that \( i \neq j \). Note that with respect to the RT0 discretization we define only faces, neither corners nor edges known from other types of substructuring. Given the decomposition, we define several additional spaces. First, we define the space \( V^{-1} \) by relaxing the condition of continuity of the normal components in the space \( V \) on \( \mathcal{F} \). More precisely, we define local spaces \( V^i \) on \( \Omega^i \), \( i = 1, \ldots, N \) by
\[ V^i = \left\{ v \in H(\Omega^i; \text{div}) : v|_K \in RT_0(K) \quad \forall K \in \mathcal{T}_h \text{ on } \Omega^i \right\}, \]
and put \( V^{-1} = V^1 \times \cdots \times V^N \). Next, we define \( Q^i = Q|_{\Omega^i} \) and the space of Lagrange multipliers as
\[ \tilde{\Lambda} = \{ \lambda \in L^2(\mathcal{F}) : \lambda = v \cdot n|_{\mathcal{F}}, \ v \in V \}. \]

**Remark 3.1.**

There are two possible approaches to a construction of the space \( V^{-1} \). The “element-wise” construction which corresponds to the standard approach to the hybrid finite element discretization, and “subdomain-wise” which is presented here and which is typically used in implementation, cf. [5, 33] for detailed discussion and comparison.
In the mixed-hybrid finite element approximation of (2)-(5) we look for \( \{ u, p, \lambda \} \in V^{-1} \times Q \times \Lambda, \) that satisfies

\[
\sum_i \left[ \int_{\Omega^i} k^{-1} u \cdot v \, dx - \int_{\Omega^i} p \nabla \cdot v \, dx + \int_{\partial\Omega^i \cap F} \lambda v \cdot n \, ds \right] = -\int_{\partial\Omega^i \cap \Gamma_N} g_N v \cdot n \, ds, \quad \forall v \in V^i, \\
- \sum_i \int_{\Omega^i} \nabla \cdot u q \, dx = -\int_{\Omega} f q \, dx, \quad \forall q \in Q^i, \\
\int_F u \cdot n \mu \, ds = 0, \quad \forall \mu \in \Lambda.
\]

Equation (10) imposes continuity condition on the normal component of the flux across \( \Gamma \) which guarantees that \( u \in V \). This condition also implies the equivalence of the two formulations (6)-(7) and (8)-(10). The Lagrange multipliers \( \lambda \) can be interpreted as the approximation to the trace of \( p \) on the \( F \), see [5] for details.

Let us now write the matrix formulation corresponding to (8)-(10) as

\[
\begin{bmatrix}
A & B^T & B_F^T \\
B & 0 & 0 \\
B_F & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u \\
p \\
\lambda
\end{bmatrix} =
\begin{bmatrix}
g \\
0 \\
0
\end{bmatrix}.
\]

(11)

It is important to note that \( A \) is block diagonal with \( N \) blocks, corresponding to subdomains \( \Omega^i, i = 1, \ldots, N, \) and each of the blocks is symmetric positive definite, cf. the first term in (8). Maryška, Rozložník and Tůma have shown in [22] that the system of equations (11) can be reduced (twice) to the Schur complement corresponding to the (subset of) Lagrange multipliers \( \lambda \) and solved efficiently by a direct or iterative solver. Here we will look at a slightly modified system compared to (11), and we will propose a preconditioner that can be applied to either original or modified system reduced to a symmetric positive definite matrix of the Schur complement with respect to the flux and pressure unknowns. To this end, let us make the following:

**Assumption 3.1.**
We are given a symmetric, positive and definite matrix

\[
C = \begin{bmatrix}
\bar{C} & C_F^T \\
C_F & \bar{C}
\end{bmatrix}.
\]

(12)

Now, let us consider system (11) modified by the matrix \( C \) and given as

\[
\begin{bmatrix}
A & B^T & B_F^T \\
B & -\bar{C} & -C_F^T \\
B_F & -C_F & -\bar{C}
\end{bmatrix}
\begin{bmatrix}
u \\
p \\
\lambda
\end{bmatrix} =
\begin{bmatrix}
g \\
\overline{f} \\
0
\end{bmatrix}.
\]

(13)

The modification (13) is motivated by the fractured porous media, discussion of which is deferred to Section 5.

Let us denote the matrix in (13) as \( \mathbb{A} \) and let us denote the Schur complement of \( A \) in \( \mathbb{A} \) by \( \mathbb{A}/A \), i.e.,

\[
\mathbb{A}/A = - \begin{bmatrix}
\bar{C} & C_F^T \\
C_F & \bar{C}
\end{bmatrix} - \begin{bmatrix} B \\
B_F \end{bmatrix} A^{-1} \begin{bmatrix} B^T \\
B_F^T \end{bmatrix}.
\]
Given Assumption 3.1 and because \( A \) is symmetric and positive definite it is easy to see, by the Sylvester law of inertia, that \( -A/A \) is also symmetric and positive definite. We note that this also holds when \( C \) is zero. Let us denote the first principal submatrix of \( -A/A \) by \( A_1 = C + BA^{-1}B^T \) and suppose that \( A_1 \) can be easily factored, e.g., in linear time similarly as the block \( A \). Then we define the second Schur complement as \( \hat{S} = ( -A/A )/A_1 \).

By the inertia argument, as before, the Schur complement \( \hat{S} \) of the positive definite matrix \( A_1 \) in the positive definite matrix \( -A/A \) is also positive definite. Next, we only refer to [22] for a detailed discussion of the fill-in and spectral properties of the matrices – in the case when \( C = 0 \). In our case the fill-in and the spectral properties depend on the properties of the matrix \( C \) which are in turn given by the structure of the fractured porous media, as we will see in Section 5, and such analysis is difficult to be performed in general. Instead, we will be interested in an efficient iterative solution, using preconditioned conjugate gradient method, of the problem

\[
\hat{S}\lambda = \hat{f},
\]

where \( \hat{f} \) is the reduced right-hand side.

4. The BDDC method

Let us define, quite similarly as the global matrix in (13), the subdomain matrices as

\[
\begin{bmatrix}
A^i & B^i_{ij} & B_{ij}^T \\
B^i & -t^iC^i_{ij} & -t^iC_{ij}^T \\
B_{ij} & -t^iC_{ij} & -t^i\tilde{C}^i
\end{bmatrix}, \quad i = 1, \ldots, N.
\]

obtained in implementation by assembling the corresponding finite element matrices. The constant \( t^i \) merely indicates that the \( C \)–block of a subdomain \( i \) is non-zero only if the subdomain boundary intersects the dimension matching interface \( F_d \subset F \), see Section 5 for details. Similarly to Assumption 3.1 we make the following:

**Assumption 4.1.**

Assume that a symmetric matrix \( C^i \) given as

\[
C^i = \begin{bmatrix}
C^i & C_{ij}^T \\
C_{ij} & \tilde{C}^i
\end{bmatrix},
\]

is either a zero, or a positive definite matrix.

Next, let \( \Lambda^i \) be defined as the space of Lagrange multipliers corresponding to \( \Gamma^i \), \( i = 1, \ldots, N \) and define a space

\[
\Lambda = \Lambda^1 \times \cdots \times \Lambda^N.
\]
The subdomain Schur complements $S^i : \Gamma^i \mapsto \Gamma^i$, for all $i = 1, \ldots, N$, can be obtained by eliminating the flux and pressure unknowns in two steps, as discussed above for the matrix in (13). The multiplication by a subdomain Schur complement can be equivalently defined, cf. [33], as: given $\lambda^i \in \Lambda^i$, determine $S^i \lambda^i$ such that

$$
\begin{bmatrix}
A^i & B^i & B^i_F \\
B^{iT} - t' C^i & -t' C^i_F & B^{iT} \\
B^{iT} - t' C^i & B^{iT} - t' C^i_F & -t' \tilde{C}^i
\end{bmatrix}
\begin{bmatrix}
u^i \\
p^i \\
\lambda^i
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
-S^i \lambda^i
\end{bmatrix}.
$$

We note that the multiplication is equivalent to solving a local Dirichlet boundary problem for a given vector $\lambda^i$, which is always well-posed, and all of the Schur complements $S^i$ are symmetric and positive definite [33].

The global Schur complement $S$ on the space $\Lambda$ is then obtained as

$$
S = \begin{bmatrix}
S^1 & & \\
& \ddots & \\
& & S^N
\end{bmatrix},
$$

and denoting by $R : \hat{\Lambda} \rightarrow \Lambda$ the injection operator, we obtain the global Schur complement $\tilde{S}$ on the space $\tilde{\Lambda}$ by

$$
\tilde{S} = R^T S R,
$$

and the space $\tilde{\Lambda}$ is such that the values of degrees of freedom from $\Lambda$ coincide on $\partial \Omega_i \cap \partial \Omega_j$ for all intersections $\Gamma^{ij}$. Since we have reduced the saddle-point problem into a problem which is symmetric and positive definite, we can apply the simple variational formulation of the BDDC method which was originally introduced in [18]. Next we formulate this approach for our model. To this end, let us write problem (14) in the variational form as

$$
s(\lambda, \mu) = f(\mu) \quad \forall \mu \in \tilde{\Lambda}.
$$

The BDDC, as a two-level preconditioner, is characterized by a selection of coarse degrees of freedom. So, we define $\tilde{\Lambda}$ as the subspace of $\Lambda$ such that the values of the coarse degrees of freedom coincide on $\mathcal{F}$ and such that

$$
\tilde{\Lambda} \subset \hat{\Lambda} \subset \Lambda.
$$

Finally, let $E : \tilde{\Lambda} \rightarrow \hat{\Lambda}$ be a projection defined by taking a weighted average, specified in Section 6, of the corresponding degrees of freedom on the substructure interfaces. With the selection of spaces and operators as above, the BDDC preconditioner $M : \hat{\Lambda}' \rightarrow \hat{\Lambda}$ can be written in the variational formulation introduced in [18] as

$$
M : r \mapsto \lambda = E \omega, \quad \omega \in \tilde{\Lambda} : \quad s(\omega, \mu) = \langle r, E \mu \rangle, \quad \forall \mu \in \tilde{\Lambda}.
$$

and the following bound of the condition number $\kappa = \lambda_{\text{max}}(M \tilde{S}) / \lambda_{\text{min}}(M \tilde{S})$ applies:
Theorem 4.1 ([33, Theorem 6.1]).

The condition number $\kappa$ is bounded by

$$\kappa \leq \delta = \sup_{\omega \in \hat{\Lambda}} \frac{\|E_\omega\|_S^2}{\|\omega\|_S^2} \leq C \left(1 + \log \frac{H}{h}\right)^2.$$ 

5. Modelling of fractures

In this section we recall from [3] the main ideas of the finite element discretization of the flow in fractured porous media, and combining meshes of different dimensions. Let us denote $\Omega_3 = \Omega$ and recall that (for simplicity) it is a polygonal domain. Next, consider that $\Omega_{d-1} \subset \Omega_d$, $d = 2, 3$, such that $\Omega_2$ consists of polygons and $\Omega_1$ consists of line segments. We will also assume that $\partial \Omega_1 \subset \partial \Omega_2 \subset \partial \Omega_3$. We decompose $\Omega_d$ into subdomains $\Omega_d^i$, $i \in I_d$ satisfying the compatibility condition

$$\Omega_{d-1} \subset F_d, \quad d = 1, 2, 3, \quad \text{where} \quad F_d = \bigcup_{i \in I_d} \partial \Omega_d^i \setminus \partial \Omega_d.$$ 

We consider equations (2)-(5) for the flux fields $u_d$ and the pressures $p_d$ on the subdomains $\Omega_d^i$, $d = 1, 2, 3$. The effective water source on a subdomain $\Omega_d^i$ is given as

$$f_d^i = \delta_d \tilde{f}_d + \sum_{j \in I_d+1} q_{d+1}^{j,i}$$

where $\delta_d$ is cross-section ($d = 1$) or thickness ($d = 2$) of the subdomain, $\tilde{f}_d$ is the density of external water sources and $q_{d+1}^{j,i}$ is the flux from subdomain $\Omega_d^{j+1}$ to subdomain $\Omega_d^i$. The fluxes also impose boundary condition of the Newton type on $\partial \Omega_{d+1}^i$ and is given by the pressure difference:

$$q_{d+1}^{j,i} = u_d^{j+1} \cdot n|_{\partial \Omega_{d+1}^i \cap \Omega_d^i} = \sigma_d (p_d^{j+1}|_{\partial \Omega_{d+1}^i \cap \Omega_d^i} - p_d|_{\Omega_d^i}).$$

the flux is zero if $\Omega_d^i$ has empty intersection with $\partial \Omega_{d+1}^j$, cf., e.g., [21] for an alternative setting.

In the following, we describe discrete mixed-hybrid formulation of the problem. The formulation and discussion of the continuous problem can be found in [3]. So, let us consider spaces

$$V^{-1} = V_1^{-1} \times V_2^{-1} \times V_3^{-1}, \quad Q = Q_1 \times Q_2 \times Q_3, \quad \hat{\Lambda} = \hat{\Lambda}_1 \times \hat{\Lambda}_2 \times \hat{\Lambda}_3,$$

where for all $d = 1, 2, 3$, we define

$$V_d^{-1} = \prod_{i \in I_d} V^i(\Omega_d^i), \quad Q_d = L^2(\Omega_d), \quad \hat{\Lambda}_d = \{ \lambda \in L^2(F_d) : \lambda = v \cdot n|_{F_d}, v \in V_d \}.$$
In the mixed-hybrid finite element approximation of the flow in fractured porous media we look for a triple
\( \{ \mathbf{u}, p, \lambda \} \in \mathbf{V}^{-1} \times Q \times \hat{\Lambda} \) that satisfies

\[
 a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) + b_F(\lambda, \mathbf{v}) = \langle g, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in \mathbf{V}^{-1}, \quad (19)
\]

\[
 b(\mathbf{u}, q) - \mathbf{c}(p, q) - c_F(\mathbf{u}, \lambda, q) = \langle \mathbf{f}, q \rangle, \quad \forall q \in P, \quad (20)
\]

\[
 b_F(\mathbf{u}, \mu) - c_F(p, \mu) - \tilde{c}(\lambda, \mu) = 0, \quad \forall \mu \in \hat{\Lambda}, \quad (21)
\]

where

\[
 a(\mathbf{u}, \mathbf{v}) = \sum_{d=1}^{3} \sum_{i \in I_d} \int_{\Omega_d^i} \frac{1}{\delta_d} k_d^{-1} \mathbf{u}_d \cdot \mathbf{v}_d \, dx,
\]

\[
 b(\mathbf{u}, q) = -\sum_{d=1}^{3} \sum_{i \in I_d} \int_{\Omega_d^i} (\nabla \cdot \mathbf{u}_d) \, q_d \, dx,
\]

\[
 b_F(\mathbf{u}, \lambda) = \sum_{d=1}^{3} \sum_{i \in I_d} \int_{\partial \Omega_d^i \cap F_d} (\mathbf{u}_d \cdot \mathbf{n}) \lambda_d \, ds,
\]

\[
 c(p, q) = 2 \sum_{d=1}^{3} \sum_{i \in I_d} \int_{\Omega_d^i} \sigma_d \, p_d \, q_d \, ds,
\]

\[
 c_F(p, \mu) = -2 \sum_{d=1}^{3} \sum_{i \in I_d} \int_{\Omega_d^i} \sigma_d \, p_d \, \mu_{d+1} \, ds,
\]

\[
 \tilde{c}(\lambda, \mu) = 2 \sum_{d=1}^{3} \sum_{i \in I_d} \int_{\Omega_d^i} \sigma_d \, \lambda_{d+1} \mu_{d+1} \, ds,
\]

\[
 \langle g, \mathbf{v} \rangle = -\sum_{d=1}^{3} \sum_{i \in I_d} \int_{\partial \Omega_d^i \cap \Gamma_N} g_N \mathbf{v} \cdot \mathbf{n} \, ds,
\]

\[
 \langle \mathbf{f}, q \rangle = -\sum_{d=1}^{3} \int_{\Omega_d} \delta_d \, f_d \, q \, dx,
\]

We have included \( \delta_d \) in the bilinear form \( a \) in order to make the system symmetric, because \( \mathbf{u}_d \) is flux (i.e., velocity times \( \delta_d \)), but Darcy’s law is for velocity. We note that (13) is related to (19)-(21) in the same ways as (11) is related to (8)-(10). Indeed, assuming that \( \delta_d \) and \( \sigma_d \) are uniformly bounded and greater than zero and that \( k_d \) is a symmetric positive definite matrix, we see that both blocks \( A \) and \( C \) in (12) and (13) are both positive definite, and Assumption 3.1 is satisfied. (bs: Do we really see that?) Indeed,

\[
 \tau(p, p) + 2c_F(p, \lambda) + \tilde{c}(\lambda, \lambda) = 2 \sum_{d=1}^{3} \sum_{i \in I_d} \int_{\Omega_d^i} \sigma_d (p_d - \lambda_{d+1})^2 \, ds
\]

and the discrete mixed-hybrid problem (13) has a unique solution.
6. The parallel solver

ToDo:

mail 1: Oni skutečně používají ten hybridní přístup na urovni prvku a skutečně zacinají od sestavení cele matice sedlobodove, pricemž bloky pro toky a tlaky nijak neprovazují prvky, tedy dokud nedojde na dimenze. Vtip je prave v tom, ze vazby mezí dimenzemi vytvářené určita provazani v bloku tlaku, takže aby mohli ty vzájemné "pretoky" mezi dimenzemi aplikovat, delají to pres tlaky. Proto není snadné implementovat to s eliminovanými toky a tlaky a rovnou sestavovat matici tuhosti pro L. multiplikatory. K tomu by nam rekli vic Honza Brezina. Moc matematicke to není, delají to pomocí jakýchsi z prstu cukaných koeeficientu pretoku kdy tok v 1D ovlivnuje tlak v 2D a podobne. Nicmene se to musí delat pred eliminací rychlosti a tlaku.

Kouzlo BDDC tkvi v tom, ze mezi domenami prirozene zajisti komunikaci v bloku Lagrangeových multiplikatorů, takže místo dvoji postupné eliminace ke konstrukci Schurov doplnku vzhledem k multiplikatorům a nasledné aplikaci DD clovek dostane tótě bez prace s explicitními Schurovými doplnky. Zda se mi to elegantní. Tohle bych videl jako jadro toho připadného clanku.

Velkou otázkou je, zda v delení na podoblasti zohlednovat dimenze (t.j. snazit se shlukovat prvky stejných dímezní k sobe) nebo jestli delat delení spis s ohledem na geometrické oblasti. Ty tri dimenze samozřejmě ve spekturu zavádejí tri hladiny vlastních císel, které mohou kazit podmíněnost... neco jako pro shell prvky v pruznosti.

mail 2: Oni udelali dvoji Schurovy doplnky a skončili u Schurova doplnku vzhledem k L. multiplikatorům. Na ne by se dala pustit BDDC a eliminovat vnitrní promené a zbude problem na rozhraní subdomen v tech L. multiplikátorech, který by se predpodmínil BDDC. Ovsem my to nedelame explicitně a BDDC resíci predchodíme tu matici v prvodním tvaru. Jelikož vazby mezi prvky a subdomenami jsou jenom mezi L. multiplikatory, BDDC si rozpozná rozhraní mezi domenami uplně stejné jako pred tím a vsecny primarní promené spadou do "interior". Na subdomenách se uziva indefinitní primy resic, ale problem, který "vidí" PCG je ekvivalentní tomu, co by se stalo, kdyby se BDDC pustila na ten 2. Schuruv doplnek. Predpokladame, ze MUMPS s eliminací nema velky problem, a ze pozna, ze blok A je blokove diagonalní.

Multilevel pak sestavi elementovou matici subdomen vzhledem k hrubym neznámym a zopakuje to, tentokrat uz jen na te SPD matici, kterou efektivne vidi. Nebo je to složitejší?

7. Numerical results

bs: it would be good if the engineering pictures had a scale (dimensions) and the numbers (in pictures) had units.
Figure 1. Example of solution to the model square problem containing only 2D elements, plot of pressure (left) and velocity vectors (right).

Figure 2. Example of solution to the model cube problem containing only 3D elements, plot of pressure (left) and velocity vectors (right).

Figure 3. Example of solution to the model cube problem containing 1D, 2D, and 3D elements, plot of pressure with fractures (left) and velocity vectors (right).
Table 1. Weak scaling test for the 2D square problem. Number of processors equals number of subdomains $N$. $n$ is the size of the global problem, $n_Γ$ is the size of the interface problem, $n_f$ is the number of faces, $n_c$ is the number of corners, PCG its. is the number of PCG iterations, and cond. est. the estimated condition number, time for solve is the sum of set-up and PCG.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$n$</th>
<th>$n/N$</th>
<th>$n_Γ$</th>
<th>$n_f$</th>
<th>$n_c$</th>
<th>its. cond.</th>
<th>time (sec)</th>
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</tr>
<tr>
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<td>26</td>
<td>9</td>
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<tr>
<td>16</td>
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<td>111k</td>
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<td>66</td>
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<td>332</td>
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Table 2. Weak scaling test for the 3D cube problem, approx. 100k unknowns per subdomain.

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<th>$n_c$</th>
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<th>time (sec)</th>
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<td>11</td>
<td>11.7</td>
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<tr>
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<td>437k</td>
<td>109k</td>
<td>2.3k</td>
<td>6</td>
<td>18</td>
<td>12</td>
<td>11.7</td>
</tr>
<tr>
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<td>118k</td>
<td>5.7k</td>
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<td>63</td>
<td>15</td>
<td>15.4</td>
</tr>
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<td>1.6M</td>
<td>103k</td>
<td>12.8k</td>
<td>56</td>
<td>168</td>
<td>16</td>
<td>12.9</td>
</tr>
<tr>
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<td>3.4M</td>
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<td>401</td>
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<td>307</td>
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<td>19</td>
<td>13.7</td>
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</tbody>
</table>

7.1. Benchmark problems

7.2. Engineering problems

Todo: (courtesy of ...) kostka regular dependence on coefficients comments: corners / no corners, two-level / multi-level,

Table 3. Weak scaling test for the 3D cube problem, approx. 200k unknowns per subdomain.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$n$</th>
<th>$n/N$</th>
<th>$n_Γ$</th>
<th>$n_f$</th>
<th>$n_c$</th>
<th>its. cond.</th>
<th>time (sec)</th>
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<tbody>
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<td></td>
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<td>set-up PCG solve</td>
<td></td>
</tr>
<tr>
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<td>210k</td>
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<td>3</td>
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<td>41.7</td>
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<td>4</td>
<td>945k</td>
<td>236k</td>
<td>3.8k</td>
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<td>18</td>
<td>14</td>
<td>51.3</td>
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<td>8.7k</td>
<td>21</td>
<td>63</td>
<td>15</td>
<td>43.3</td>
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<tr>
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<td>21.4k</td>
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<td>162</td>
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<td>46.0</td>
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<td>6.1M</td>
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<td>409</td>
<td>20</td>
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<td>939</td>
<td>23</td>
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</table>
Table 4. Weak scaling test for the cube problem with 1D, 2D, and 3D elements, approx. 200k unknowns per subdomain.

<table>
<thead>
<tr>
<th>N</th>
<th>n/N</th>
<th>n_f</th>
<th>n_c</th>
<th>its. cond.</th>
<th>time (sec)</th>
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<tr>
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<td>set-up</td>
<td>PCG solve</td>
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<td>417k</td>
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<td>4</td>
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<td>203k</td>
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<td>19 5.99</td>
<td>35.9 8.6 44.6</td>
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<td>10.6k</td>
<td>33 18.41</td>
<td>58.9 16.1 75.0</td>
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<tr>
<td>16</td>
<td>3.7M</td>
<td>229k</td>
<td>23.9k</td>
<td>37 22.68</td>
<td>66.1 19.0 85.1</td>
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<td>32</td>
<td>6.7M</td>
<td>209k</td>
<td>46.1k</td>
<td>49 195.18</td>
<td>66.7 26.4 93.2</td>
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<td>64</td>
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<td>203k</td>
<td>163.5k</td>
<td>80 37.63k</td>
<td>70.8 49.5 120.3</td>
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</table>

Figure 4. Weak scaling test for the 2D square problem (left), and the 3D cube problem (right), approx. 100k unknowns per core.

Acknowledgements

The research has been supported by ... B. Sousedík gratefully acknowledges support from the DOE/ASCR and the NSF PetaApps award number 0904754.

Table 5. Strong scaling test for the problem of the Melechov locality containing 2D and 3D elements, size of the global problem is n = 1.5M unknowns.

<table>
<thead>
<tr>
<th>N</th>
<th>n/N</th>
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<th>n_c</th>
<th>its. cond.</th>
<th>time (sec)</th>
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<tr>
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<td></td>
<td>set-up</td>
<td>PCG solve</td>
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<tr>
<td>8</td>
<td>184k</td>
<td>5k</td>
<td>42</td>
<td>54 63.3</td>
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<td>92k</td>
<td>7k</td>
<td>105</td>
<td>54 62.4</td>
<td>5.1 12.5 17.6</td>
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<td>46k</td>
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<td>244</td>
<td>77 918.5</td>
<td>2.3 8.9 11.3</td>
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<td>18k</td>
<td>608</td>
<td>75 235.7</td>
<td>1.3 4.6 5.9</td>
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<td>71 485.7</td>
<td>0.7 2.4 3.1</td>
</tr>
<tr>
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<td>33k</td>
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<td>63 169.0</td>
<td>0.4 1.9 2.3</td>
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<tr>
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<td>7657</td>
<td>79 3241.4</td>
<td>0.5 3.5 4.1</td>
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Figure 5. Weak scaling test for the 3D cube problem (left), and for the cube problem with 1D, 2D, and 3D elements (right), approx. 200k unknowns per core.

Figure 6. Solution of the problem of the Melechov locality containing 2D and 3D elements, plot of pressure with fractures (left) and velocity vectors (right), mesh contains 210k elements.

References


Table 6. Strong scaling test for the problem of the Melechov locality containing 2D and 3D elements, size of the global problem is $n = 15M$ unknowns.

<table>
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<td>108</td>
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</table>
Figure 7. Strong scaling test for the problem of the Melechov locality containing 2D and 3D elements and 1.5M unknowns, computational time (left) and speed-up (right) separately for set-up and PCG phases, and their sum (total).

Figure 8. Strong scaling test for the problem of the Melechov locality containing 2D and 3D elements and 15M unknowns, computational time (left) and speed-up (right) separately for set-up and PCG phases, and their sum (total).


Table 7. Strong scaling test for the problem of the Bedřichov tunnel containing 2D and 3D elements, size of the global problem is $n = 7.8$M unknowns.

<table>
<thead>
<tr>
<th>$N$</th>
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<tbody>
<tr>
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<tr>
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</table>
Figure 9. Strong scaling test for the problem of the Bedřichov tunnel containing 2D and 3D elements and 15M unknowns, computational time (left) and speed-up (right) separately for set-up and PCG phases, and their sum (total).


